

PROPERTIES OF M-GROUPOIDS RELATED TO (M,N)-VARIETIES

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Abstract

An algebra (A, F) is said to be an m -groupoid if F consists of a single m -ary operation. In this paper, the results of the research on such algebraic structures are presented, regarding some special classes of m -groupoids. The main ones are closely connected to varieties with the property (m,n) . Every member of an (m,n) -variety of m -groupoids is an m -quasigroup, for $m < n$.

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INTRODUCTION

An algebra (A, F) is said to be an m -groupoid if F consists of only one m -ary operation, $m \geq 1$. This means that $F = \{f\}$ and the rank of f is $m \geq 1$, $ar(f) = m$. We do not consider the case of $m = 0$, i.e. algebras with one nullary operation, which is actually a set with a constant (one fixed element). Since $F = \{f\}$ is one-element set, we denote such algebra by (A, f) .

For 2-groupoid, we just say that it is a groupoid, and usually its operation is denoted multiplicatively. According to the introduced notion, a mono-unary algebra is 1-groupoid, and it can be considered as a transformation of a set.

An m -groupoid (Q, f) is said to be *quasigroup* if it satisfies the following property: any m elements of arbitrary chosen $m + 1$ elements $a_1, a_2, \dots, a_m, a_{m+1}$ of Q uniquely determine the remaining element in the equation $f(a_1, a_2, \dots, a_m) = a_{m+1}$. We use the term m -quasigroup to emphasize the rank of the operation, except in case of binary operation, i.e. (ordinary) quasigroups.

An m -groupoid (Q, f) is said to be *cancellative* if $f(a_1, a_2, \dots, a_i, x, a_{i+2}, \dots, a_m) = f(a_1, a_2, \dots, a_i, y, a_{i+2}, \dots, a_m) \Rightarrow x = y$, for every $i = 0, 1, \dots, m - 1$ and each $x, y, a_j \in Q$, $j \in \{1, 2, \dots, m\} \setminus \{i + 1\}$.

An m -groupoid (Q, f) is *solvable* if the equation $f(a_1, a_2, \dots, a_i, x, a_{i+2}, \dots, a_m) = a$ has a solution on the variable x , for every $i = 0, 1, \dots, m - 1$ and $a_j \in Q$, $j \in \{1, 2, \dots, m\} \setminus \{i + 1\}$.

1. EQUIVALENCE OF SOME PROPERTIES OF M-GROUPOIDS

The stated definitions in the first section imply that m -groupoid is m -quasigroup if and only if it is cancellative and solvable.

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The well known property for (ordinary) groupoids holds for arbitrary m -groupoid. Its proof is given in the sequel, and it will be used for the proofs of the main results.

Lemma 1: Let (Q, f) be a finite m -groupoid. Then the following statements are equivalent:

- (i) (Q, f) is m -quasigroup;
- (ii) (Q, f) is cancellative m -groupoid;
- (iii) (Q, f) is solvable m -groupoid.

Proof: Let $Q = \{q_1, q_2, \dots, q_n\}$.

(i) \Rightarrow (ii): This implication follows directly from the definitions. Namely, let $f(a_1, a_2, \dots, a_i, x, a_{i+2}, \dots, a_m) = a_{m+1} = f(a_1, a_2, \dots, a_i, y, a_{i+2}, \dots, a_m)$, $0 \leq i < m$. Since (Q, f) is m -quasigroup, the element a_{i+1} is uniquely determined by the elements a_j , $j = 1, 2, \dots, m+1, j \neq i+1$ in the equation $f(a_1, \dots, a_i, a_{i+1}, a_{i+2}, \dots, a_m) = a_{m+1}$. At the same time, this means that $x = y$.

(ii) \Rightarrow (iii): Consider $f(a_1, a_2, \dots, a_i, x, a_{i+2}, \dots, a_m) = a_{m+1}$, $i \in \{0, 1, \dots, m-1\}$, and $a_j \in Q$, $j \in \{1, 2, \dots, m+1\} \setminus \{i+1\}$. Let $P \subseteq Q$ be determined by $P = \{f(a_1, a_2, \dots, a_i, q_j, a_{i+2}, \dots, a_m) \mid j \in \{1, 2, \dots, n\}\}$.

By $f(a_1, a_2, \dots, a_i, x, a_{i+2}, \dots, a_m) = f(a_1, a_2, \dots, a_i, y, a_{i+2}, \dots, a_m) \Rightarrow x = y$, we get that $P = Q$, which means that there is $k \in \{1, 2, \dots, n\}$, which satisfies $a_{m+1} = f(a_1, a_2, \dots, a_i, q_k, a_{i+2}, \dots, a_m)$.

(iii) \Rightarrow (i): In order to prove that (Q, f) is m -quasigroup, it suffices to show that the equation $f(a_1, a_2, \dots, a_i, x, a_{i+2}, \dots, a_m) = a_{m+1}$ has a solution x which is unique (since (Q, f) is algebra, so $f(a_1, a_2, \dots, a_i, x, a_{i+2}, \dots, a_m)$ is uniquely determined). The m -groupoid (Q, f) is solvable, and therefore there is an element $q \in Q$, such that $f(a_1, a_2, \dots, a_i, q, a_{i+2}, \dots, a_m) = a_{m+1}$. Let us assume that there is another element $p \in Q$, $p \neq q$, such that $f(a_1, a_2, \dots, a_i, p, a_{i+2}, \dots, a_m) = a_{m+1}$. Then, for the set P defined as before, we obtain $|P| < n$, implying that there is an element $a \in Q \setminus P$ such that $a \neq f(a_1, a_2, \dots, a_i, x, a_{i+2}, \dots, a_m)$, for every $x \in Q$. The latest contradicts the fact that (Q, f) is solvable m -groupoid. \square

Let (A, f) be an m -groupoid. Then every subalgebra (B, f) of (A, f) is called an m -subgroupoid of (A, f) . If (B, f) is m -quasigroup, we say that it is m -subquasigroup of (A, f) .

The preceding lemma shows that sufficient condition in order cancellative m -groupoids to be m -quasigroups is their finiteness. The following property will show that this is also the case when some other conditions are fulfilled, and it is a kind of generalization. Namely, it suffices only the finitely generated subalgebras of an m -groupoid to be finite, moreover, only those which are generated by m elements.

Lemma 2: Let (A, f) be cancellative m -groupoid and every subalgebra of (A, f) generated by m elements be finite. Then (A, f) is m -quasigroup.

Proof: Since (A, f) is cancellative m -groupoid, it suffices to show that any equation $f(a_1, a_2, \dots, a_{i-1}, x, a_{i+1}, \dots, a_m) = a_{m+1}$ has a solution x . Let B denotes the m -subgroupoid which is generated by the elements $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m, a_{m+1}$. It is both finite and m -cancellative, so the given equation has a solution a in B , according to Lemma 1. Then a is also a solution in A . \square

2. (M, N)-VARIETIES AND M-GROUPOIDS

The stated results in the previous section were obtained as a part of the research on (m, n) -algebras. In this section, the connection between varieties of such algebras and m -groupoids is presented.

Let m and n be positive integers, $m \leq n$. An algebra A is said to have the property (m, n) , or that is an (m, n) -algebra, if every subalgebra of A which is generated by m -element subset has exactly n elements. A class of algebras is said to have the property (m, n) , or that it is an (m, n) -class, if each of its algebras is (m, n) -algebra.

A variety of algebras (equational class) is the class of all algebras of a given signature (type) satisfying some set of identities. It is well known (Birkoff's theorem) that a class of algebraic structures of the same signature is a variety if and only if it is closed under the taking of direct products, homomorphic images, and subalgebras. Here we will use the name (m, n) -variety to denote a variety with the property (m, n) , i.e. a variety which is an (m, n) -class of algebras. Every algebra (A, f) of an (m, n) -variety, $m < n$, satisfies the idempotent law, that is $f(a, a, \dots, a) = a$, for every element $a \in A$, and arbitrary operation $f \in F$.

Theorem 1: Let V be an (m, n) -variety of m -groupoids, and $m < n$. Then every algebra of V is m -quasigroup.

Proof: Let (A, f) be an algebra of V . First we consider the case of algebras with at least m elements, so let $|A| \geq m$. Since the subalgebras of A generated by m elements have n elements, it is sufficient to prove that A is cancellative m -groupoid, by Lemma 2.

Let us assume that there exist elements $a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_m, b, c \in A$, such that $f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_m) = f(a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_m)$, and $b \neq c$. If all of the elements $a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_m, b, c$ are distinct, then let A_1 be the subalgebra of A generated by the set $\{a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_m, b\}$, otherwise let A_1 be the subalgebra of A generated by some set $P \subseteq A$, such that $|P| = m$, and $a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_m, b, c \in P$. Then, since A is (m, n) -algebra, we have that $|A_1| = n$, so let $A_1 = \{x_1, x_2, \dots, x_n\}$.

Consider the subalgebra A_2 of $A \times A$, which is generated by the set $\{(x, x) \mid x \in P\}$. It has n elements, $(x_1, x_1), (x_2, x_2), \dots, (x_n, x_n)$. Let $y = f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_m)$, and note that $(y, y) \in A_2$. Let A_3 be the subalgebra of $A \times A$, which is generated by some set Q in which the following holds: $\{(y, y)\} \cup \{(a_j, a_j) \mid j = 1, \dots, i-1, i+1, \dots, m\} \subseteq Q \subseteq A_2$, and $|Q| = m$. Then, since any subalgebra of an (m, n) -algebra which is generated by m distinct elements is also generated by any of its m -element subsets, we obtain that $A_2 = A_3$. By the same argument, the subalgebra A_4 of $A \times A$, generated by the set $(Q \setminus \{(y, y)\}) \cup \{(b, c)\}$ contains the element (y, y) , since (y, y) is the element which is the result of the application of the operation f to the m -tuple $((a_1, a_1), \dots, (a_{i-1}, a_{i-1}), (b, c), (a_{i+1}, a_{i+1}), \dots, (a_m, a_m))$. In this way, we get $A_2 = A_4$. But $A_2 = \{(x_j, x_j) \mid j = 1, 2, \dots, n\}$ and $(b, c) \in A_4 = A_2$, $b \neq c$, which is a contradiction.

Now, let $1 < |A| < m$, and assume that for some elements of the set A , the following holds: $f(a_1, \dots, a_i, a, a_{i+2}, \dots, a_m) = f(a_1, \dots, a_i, b, a_{i+2}, \dots, a_m) = c$, and $a \neq b$. Let U be an algebra of V which is generated by m elements. Then $A \times U$ is in V , and by the first part of the proof, $A \times U$ is an m -quasigroup. But then, for arbitrary $d \in U$, since any algebra of (m, n) -variety is idempotent, we get the following equalities: $f((a_1, d), \dots, (a_i, d), (a, d), (a_{i+2}, d), \dots, (a_m, d)) = (c, d) = f((a_1, d), \dots, (a_i, d), (b, d), (a_{i+2}, d), \dots, (a_m, d))$, and $(a, d) \neq (b, d)$, contradicting the fact that $A \times U$ is m -quasigroup. \square

Corollary: Let V be a $(2, n)$ -variety of groupoids, $2 < n$, and A be a groupoid of V . Then A is a quasigroup. \square

The research yielded an interesting result in the special case of $n = m + 1$.

Theorem 2: Let V be nontrivial $(m, m + 1)$ -variety of m -groupoids. Then

$$f(x_1, x_2, \dots, x_m) \approx f(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_m))$$

is identity in V , for arbitrary permutation α of $\{x_1, x_2, \dots, x_m\}$, $m \in \{2, 3\}$, and in the case of $m = 3$, $f(x, x, y) \approx y$ is identity in V .

Proof: Let U be the free V -algebra over an m -element set $\{x_1, x_2, \dots, x_m\}$. Then $U = \{x_1, x_2, \dots, x_m, f(x_1, x_2, \dots, x_m)\}$, since U is $(m, m + 1)$ -algebra, which is also an m -quasigroup. Namely, the assumption $f(x_1, x_2, \dots, x_m) = x_i$, for some $i \in \{1, 2, \dots, m\}$, implies the identity $f(x_1, x_2, \dots, x_m) \approx x_i$ in V . Then, for every non-trivial V -algebra A , and $x, y \in A$, where $x \neq y$, in the case of $i \neq 1$ we obtain $f(x, y, y, \dots, y) = y = f(y, y, y, \dots, y)$, and in the case of $i = 1$ we get $f(y, y, \dots, y, x) = y = f(y, y, \dots, y, y)$. In both cases, there is a contradiction to the fact that A is an m -quasigroup.

Similarly, $f(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_m)) \neq x_i$, for any permutation α and every $i \in \{1, 2, \dots, m\}$. Therefore, $f(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_m)) = f(x_1, x_2, \dots, x_m)$.

Note that $f(x_1, x_2, x_3, \dots, x_m) \neq f(x_2, x_2, x_3, \dots, x_m)$. In that way, we conclude that there exists $i \in \{2, 3, \dots, m\}$, such that $f(x_2, x_2, x_3, \dots, x_m) = x_i$ (1)

($i \neq 1$ since V is non-trivial variety). If $m = 3$, then $f(x_2, x_2, x_3) = x_3$, since $f(x_2, x_2, x_2) = x_2$ and U is 3-quasigroup.

Let $m \geq 4$ and assume that $i = 2$ in (1). Then $f(x, x, \dots, x, y) \approx x$ is identity of V , which is impossible since V is a variety of idempotent m -quasigroups. Therefore $i \neq 2$. By similar argumentation, $i \neq j$ in (1), for every $j \geq 3$. \square

The defining identities for varieties with the property (2,3) and (3,4) and their properties are well known. Their members are algebraic representations of Steiner triple and Steiner quadruple systems, correspondingly. Unfortunately, analogous construction of $(m, m + 1)$ -varieties of m -quasigroups, for $m \geq 4$ can't be done, according to the proof of Theorem 2.

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