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ENDOWMENT LIFE INSURANCE

Zeljko Sain[1]
Jasmina Selimovic

Abstract:
The aim of the paper that treats the actuarial model of insurance in case of survival or early death is to show the actuarial methods and methodology for creating a model and an appropriate number of sub-models of the most popular form of life insurance in the world. The paper applies the scientific methodology of the deductive character based on scientific, theoretical knowledge and practical realities. Following the basic theoretical model's determinants, which are at the beginning of the paper, the basic difference between models further in this paper was carried out according to the character of the premium to be paid. Finally, the financial repercussions of some models are presented at examples in insurance companies. The result of this paper is to show the spectrum of possible forms of capital endowment insurance which can be, without major problems, depending on the financial policy of the company, applied in actual practice. The conclusion of this paper shows the theoretical and the practical reality of this model, life insurance, and its quantitative and qualitative guidelines.

Keywords: capital endowment insurance, life insurance, actuarial mathematics of life insurance, the insured, the insurer, the risk in life insurance.

Jel Classification: G22

INTRODUCTION

Life insurance is a historical category. It has passed the unknown number of its evolutionary changes in a number of revolutionary transformations: from primitive to modern forms of social market forms of personal and interdisciplinary character. One form of life insurance and insurance lump-sum payments to the insured person in case of survival of the agreed year of life or even to ensure the event of early death of the insured person. This model of life insurance in actuarial mathematics is called capital endowment insurance. It should be emphasized that this is the most popular and most common form of life insurance in the world. It is quite logical because in this form of insurance contracts, insurance company is obliged to provide payment for the insured person or for beneficiary of insurance if the insured event occurs within the agreed timeframe.

[1] **Zeljko Sain**, Ph.D., Associate Professor; **Jasmina Selimovic**, Ph.D., Assistant Professor, School of economics and business in Sarajevo, Bosnia and Herzegovina.

The risk in this model of life insurance is primarily related to the short life of the person who is insured. However, the short life of the insured person also means the period of payment that does not last long enough to get out of premium payments the actuarial reserve, mathematical provision, which will constitute a fund for the payment of the sum insured. Also, there is another uncertainty in variable forms of insurance of capital: how much will be a minimum amount of capital that should be paid at the time of its payment to the insurance beneficiary.[2]

MODELS AND SUBMODELS OF CAPITAL ENDOWMENT INSURANCE

To be able to specify models and sub models of capital endowment insurance it is necessary to determine the criteria for possible division of the synthetic forms of life insurance. There could be many criteria for the division, but following are relevant ones: the character of lump-sum payments (capital) and the character of payments (premiums).

In addition to the special character of the payment, which can be repeated, it is possible to distinguish individual insurance in case of death or survival, as well as there is the possibility for the contract to specify the moment of their creation.

According to the first criterion, the character of the lump-sum payments, the capital endowment insurance model can be differentiated:

- payment of the fixed amount (constant capital)
- payment of the variable amount (variable capital).

Related to paying premiums, it is possible to distinguish single premiums and multiple premiums (equal and/or variable, in one or more series).

Variability in actuarial science is based on exactly defined mathematical laws (which can be recognized in the form of arithmetic and/or a geometric progression). Variability can have upward and downward trend and thus directly affect the amount and flow of cash flows in this model and sub models of life insurance.

Mentioned sub models are usually defined by premium payments (single or multiple[3]), by possible conversion of insurance (which is the result of capitalization of insurance or determining the surrender value in insurance) and duration of contract relations.

It is obvious that all the indicated features give enough space for the creation of numerous sub models within two basic models of endowment insurance.[4] It is natural that the basic principle of actuarial mathematics is involved — the principle of equivalence of deposits and withdrawals at any time point within the obligation period.

[2] This work will be considered only a nominal amount of capital, and not the real size. So, following will not be considered (a) the possible effects of inflation or deflation in the real value of minimal amount of capital, and (b) possibility of participation in profit of insurance beneficiaries or insured persons profit insurers in the investments of funds.

[3] When thinking of premium it should be considered: the possibility of paying a premium in many series, the character of the payments (the premiums may be the same throughout the contractual period or variable), the frequency of premium payments, and possible changes in tables that are used by insurers when quantify risk (which is represented by premium).

[4] The nature and purpose of this study inevitably limit its further explicitly presented content.

ACTUARIAL TECHNICAL BASE OF ENDOWMENT INSURANCE

The character of insurance defines the pricing of the insured risk when inevitably observing a large number of cases from the past as necessary. Based on data obtained from such observation, possibility of unexpected events is reduced, or individual case is starting to behave according to certain rules. This brings us to the basic natural law - the law of large numbers. The law of large numbers together with the statistical probability theory is statistical base for proper actuarial quantifying mortality risk. The risk of mortality is presented in mortality tables, which are only partly applicable in the area of quantifying the risk of mortality. Namely, in order for the data on mortality to be adequate and useful in assessing the mortality risk (and hence used in the creation of the aforementioned models of endowment insurance), it is necessary to adapt data on mortality to this purpose, which implied their presentation in form of actuarial tables. Presentation of data in the actuarial tables (by using commutation symbols) means using a discount rate for graduated data on mortality.[5]

Given the basic characteristics of the set model of life insurance, which are mostly determined by the nature of the payment of insured amount, which means by event or contingency, it is necessary to consider the commutation symbols of dead and living persons (insured sum payment is conditioned with endowment of the age defined by contract or earlier death of the insured person).

In accordance with the basic settings of actuarial science, commutation symbols living people are defined in three levels. The lowest is defined as $D_x = l_x \cdot v^x$ (where l_x denotes number of living persons aged x years, and v denotes a discount factor). For the purposes of the model that follows, commutation symbols of higher rand can be derived as follows:
$N_x = D_x + D_{x+1} + D_{x+2} + D_{x+3} + \cdots$ and $S_x = N_x + N_{x+1} + N_{x+2} + N_{x+3} + \cdots$ (Sain 2009, 33).

If in the observed model sum insured is being paid in case of death of a person, it is necessary, according to the number of persons in the tables and the number of deceased persons, to develop commutation symbols for deceased persons as follows $C_x = d_x \cdot v^{x+1}$ (where d_x denotes number of deaths of persons aged x years). Previous analogy, and for models whose deriving follows commutation symbols of higher rank for the deceased persons would be $M_x = C_x + C_{x+1} + C_{x+2} + C_{x+3} + \cdots$ and $R_x = M_x + M_{x+1} + M_{x+2} + M_{x+3} + \cdots$(Sain 2009, 34).

MODEL OF CAPITAL ENDOWMENT INSURANCE AND SUB MODELS

Single premiums

As stated earlier, in line with theoretical assumptions, it is possible to develop a range of models. The authors of this paper decided to perform only those models where the sum insured is paid as a constant amount throughout the period obligations.

In order to provide a one-time payment amount (lump-sum payment) in case the insured survives until age defined by contract, it is necessary that the insured or any

[5] Data on mortality are obtained by population census (using longitudinal or transverse method). Special techniques done graduating raw data on mortality and probability, and these data are the base for calculating commutation symbols in actuarial tables.

other legal or natural person on his/her behalf (the policyholder), pays the defined premium. The premium level is determined by insurance company, while the frequency of payment is determined by the insured or policyholder. Payments may be single or multiple. Depending on the frequency of payments, the amount of the premium will be determined. It should be said that each of the premium payment modes implies some financial repercussions as for the insurers and the insured as well.

This model can be determined as follows: person aged x years would like to conclude a contract on endowment insurance with insurance company. Such an agreement would entail a one-time payment of amount K in case that insured person lives until $x+n$ years or in case that insured person dies earlier, provided that death occurs during the period:

a) from age x to x + n and
b) from $x+k$ to $x+k+n$ (where the period of delay in payment of the sum insured in respect of payment of premiums is denoted as k).

For this insurance contract policyholder (or insured) pays a one-time premium at the beginning of contract period ($EA_{xn]}$). It is necessary to define the actuarial model, which will quantify the mortality risk for this model of life insurance.

This sub model has two different payment periods, which directly affect all of its essential elements.

a) model when the period of payment takes n years

Bearing in mind the basic principle of actuarial mathematics — the principle of equivalence, this sub model can thus be present mathematically:

$$EA_{xn]} \cdot l_x = K \cdot d_x \cdot v + K \cdot d_{x+1} \cdot v^2 + K \cdot d_{x+2} \cdot v^3 + \cdots + K \cdot d_{x+n-1} \cdot v^n + K \cdot l_{x+n} \cdot v^n$$
$$EA_{xn]} \cdot D_x = K \cdot (C_x + C_{x+1} + C_{x+2} + \cdots + C_{x+n-1} + D_{x+n})$$
$$EA_{xn]} = K \cdot \frac{M_x - M_{x+n} + D_{x+n}}{D_x} \qquad \text{Relation (1)}$$

b) model when the period of payment lasts $n-k$ years

Bearing in mind the basic principle of actuarial mathematics — the principle of equivalence, this sub model can thus be present mathematically:

$$EA_{xn]} \cdot l_x = K \cdot d_{x+k} \cdot v^{k+1} + K \cdot d_{x+k+1} \cdot v^{k+2} + \cdots + K \cdot d_{x+k+n-1} \cdot v^{k+n} + K \cdot l_{x+k+n} \cdot v^{k+n}$$
$$EA_{xn]} \cdot D_x = K \cdot (C_{x+k} + C_{x+k+1} + \cdots + C_{x+k+n-1} + D_{x+k+n})$$
$$k/EA_{xn]} = K \cdot \frac{M_{x+k} - M_{x+k+n} + D_{x+k+n}}{D_x} \qquad \text{Relation (2)}$$

Multiple premiums

If the policyholder (or insured) decides to pay premiums annually or regularly (semiannual, monthly or similarly), then, according to the actuarial science laws, the amount of premiums is being determined in a different way than is the case with models that were previously mentioned. Specifically, from the financial point of view, this payment method defines cash flows and income to the insurer (and those have a different treatment than it is the case with single premium payments, where the influx of all funds is single and happens at the beginning of contract period). Thus the cash flows affect the creation of the fund for the payment of the insured sum.

Regular (recurring, multiple) premiums can be paid in one or more series, in constant or variable amounts, in annual or shorter cycles, at the beginning of the period (in advance) or at the end of the period (arrear). In this paper when defining the basic model, three criteria were considered, namely: the period of premium payment, period of payment of the insured amount and duration of contract relations. Given the aforementioned criteria below are quantified actuarial risks for five sub models initial shape that can have a large number of special forms (in this text further definitions of sub models will not be elaborated). Initial sub models are:

Criterion	Model 1	Model 2	Model 3	Model 4	Model 5
Period of premium payment	$n_u = n$	$n_u = n$	$n_u < n$	$n_u < n$	$n_u < n$
Period of insured sum payment	n_i	$n_i < n$	$n_i = n$	$n_i < n$	$n_i < n$
Contract duration	n	n	n	n	n

1. periods of premium payment and payment of the insured capital are equal and corresponding obligations period (contract duration),
2. periods of premium payment are the same as contract duration and period of the insured capital payment is shorter than the period of contract,
3. premium payment period is shorter than the period of contract and payment of the insured capital is equal to the period of contractual obligations,
4. period of premium payment and period of insured capital payment are shorter than the period of contract, and payment of capital directly follows the period of payment of premium,
5. same as under the number fourth only the period of payment of the insured capital does not follow the period of premiums payment.

In all these forms of sub models initial premiums may be differently arranged. In fact, everything depends on the actual circumstances in which there are parties.

Ad 1.Sub model where the period of premium payment and payment of the insured capital are equal and corresponding obligations period can be determined as follows:

Person of age x years wants to provide a one-time payment of amount K in case of survival until $x + n$ years or in case of earlier death, if the premiums are being paid during the entire period of contract as well as insured capital. What is the net premium for this insurance plan?

$P_1, P_2, P_3, P_4, \dots, P_n$ – premiums that can be:
$P(EA_{xn]})$ and /or $P^{(m)}(EA_{xn]})$ and /or $P_{(1)}(EA_{xn]})$ and /or $P_{(1)}^{(m)}(EA_{xn]})$

a) If premiums are $P(EA_{xn]})$, then the principle of equivalence of deposits and withdrawals for this sub model will look like this:

$$\ddot{a}_{xn]} \cdot l_x = l_x + l_{x+1} \cdot v + l_{x+2} \cdot v^2 + l_{x+3} \cdot v^3 + \cdots + l_{x+n-1} \cdot v^{n-1}$$
$$\ddot{a}_{xn]} \cdot l_x \cdot v^x = l_x \cdot v^x + l_{x+1} \cdot v^{x+1} + l_{x+2} \cdot v^{x+2} + l_{x+3} \cdot v^{x+3} + \cdots + l_{x+n-1} \cdot v^{x+n-1} \ddot{a}_{xn]} \cdot D_x$$
$$= D_x + D_{x+1} + D_{x+2} + D_{x+3} + \cdots + D_{x+n-1}$$
$$\ddot{a}_{xn]} \cdot D_x = N_x - N_{x+n}$$

Given the nature of the model, the symbol $\ddot{a}_{xn]}$ is appropriate to be replaced with P. Mathematical logic in all previously derived models, namely with the application of the equivalence principle in a way that all payments are discounted at a initial moment, and then multiplying the obtained equation with v^x, we get the expression:

$$EA_{xn]} \cdot l_x \cdot v^x = K \cdot (d_x \cdot v^x + d_{x+1} \cdot v^{x+2} + \cdots + d_{x+n-1} \cdot v^{x+n} + l_{x+n} \cdot v^{x+n})$$

Equalization of payments and disbursements, we obtain the model with following characteristics: premiums are paid multiple times and in yearly intervals, at the beginning of each year, throughout the duration of contract relations; premiums are equal amount in each year and provide a lump sum of capital to be paid at the end year in which the insured person dies or at the end of the year in which the insured survives age $x+n$.

$$P(EA_{xn\rceil})\left(\frac{N_x-N_{x+n}}{D_x}\right)=K\cdot\frac{M_x-M_{x+n}+D_{x+n}}{D_x}$$

$$P(EA_{xn\rceil})=K\cdot\frac{M_x-M_{x+n}+D_{x+n}}{N_x-N_{x+n}} \qquad \text{Relation (3)}$$

b) If premiums are $P^{(m)}(EA_{xn\rceil})$, then the principle of equivalence of deposits and withdrawals for this sub model will look like this:

$$\begin{aligned}\ddot{a}_{xn\rceil}^{(m)}\cdot l_x &= l_x+l_{x+1}\cdot v+l_{x+2}\cdot v^2+\cdots+l_{x+n-1}\cdot v^{n-1}+\\ &+l_{x+\frac{1}{m}}\cdot v^{\frac{1}{m}}+l_{x+1\frac{1}{m}}\cdot v^{1\frac{1}{m}}+l_{x+2\frac{1}{m}}\cdot v^{2\frac{1}{m}}+\cdots+l_{x+n-\frac{m-1}{m}}\cdot v^{n-\frac{m-1}{m}}+\\ &+l_{x+\frac{2}{m}}\cdot v^{\frac{2}{m}}+l_{x+1\frac{2}{m}}\cdot v^{1\frac{2}{m}}+l_{x+2\frac{2}{m}}\cdot v^{2\frac{2}{m}}+\cdots+l_{x+n-\frac{m-2}{m}}\cdot v^{n-\frac{m-2}{m}}+\cdots+\\ &+l_{x+\frac{m-1}{m}}\cdot v^{\frac{m-1}{m}}+l_{x+1\frac{m-1}{m}}\cdot v^{1\frac{m-1}{m}}+l_{x+2\frac{m-1}{m}}\cdot v^{2\frac{m-1}{m}}+\cdots+l_{x+n-\frac{1}{m}}\cdot v^{n-\frac{1}{m}}\end{aligned}$$

$$\begin{aligned}\ddot{a}_{xn\rceil}^{(m)}\cdot D_x &= N_x-N_{x+n}\\ &+N_{x+\frac{1}{m}}-N_{x+n+\frac{1}{m}}\\ &+N_{x+\frac{2}{m}}-N_{x+n+\frac{2}{m}}+\cdots+\\ &+N_{x+\frac{m-1}{m}}-N_{x+n+\frac{m-1}{m}}\end{aligned}$$

On the basis of mutual relations between commutation symbols and after appropriate mathematical ordering it is possible to get the final relation for this model of premiums payment to be equated with a previously derived model for one-time insurance payments. Such a mathematical treatment will result with actuarial model of the following characteristics: multiple premiums are paid and in m intervals in one year, at the beginning of each period, throughout the duration of contract relations; premiums are of equal amount in each year and provide a lump sum of capital that will be paid at the end of the year in which the insured person dies or at the end of the year in which the insured survives age $x+n$.

$$P^{(m)}(EA_{xn\rceil})\left[\frac{m(N_x-N_{x+n})+\frac{m-1}{2}D_{x+n}}{D_x}-\frac{m-1}{2}\right]=K\cdot\frac{M_x-M_{x+n}+D_{x+n}}{D_x}$$

$$P^{(m)}(EA_{xn\rceil})=\frac{K\cdot\dfrac{M_x-M_{x+n}+D_{x+n}}{D_x}}{\dfrac{m(N_x-N_{x+n})+\dfrac{m-1}{2}D_{x+n}}{D_x}-\dfrac{m-1}{2}}$$

Relation (4)

c) If the premiums are $P_{(1)}(EA_{xn\rceil})$, then the principle of equivalence of deposits and withdrawals for this sub model will look like this:

i. If the premium is behaving in arithmetic progression:

$$\begin{aligned}\ddot{A}_{xn\rceil}\cdot l_x &= R_1\cdot(l_x+l_{x+1}\cdot v+l_{x+2}\cdot v^2+\cdots+l_{x+n-1}\cdot v^{n-1})\pm\\ &\pm d\cdot(l_{x+1}\cdot v+2\cdot l_{x+2}\cdot v^{x+2}+\cdots+(n-1)\cdot l_{x+n-1}\cdot v^{n-1})\end{aligned}$$

$$\ddot{A}_{xn\rceil}\cdot D_x=R_1\cdot(D_x+D_{x+1}+D_{x+2}+\cdots+D_{x+n-1})\pm d\cdot(D_{x+1}+2\cdot D_{x+2}+\cdots+(n-1)\cdot D_{x+n-1})$$

After putting the mathematical result is the final expression for calculating the amount of premiums in the model, model has the following characteristics: multiple premiums are paid in yearly intervals, at the beginning of each year, throughout the duration of contract relations; the premium varies throughout the obligations in such a way that in each successive period the premium is higher or lower than premiums in the previous period for the same amount of money and they provide one capital payment to be paid at the end of the year in which the insured person dies or at the end of the year in which the insured survives age $x+n$. Defined premiums, according to the principle of equivalence, equate with the withdrawals so the expression follows:

$$\frac{P_1 \cdot (N_x - N_{x+n}) \pm d \cdot [S_{x+1} - S_{x+n} - (n-1) \cdot N_{x+n}]}{D_x} = K \cdot \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

$$P_1 = \frac{K \cdot (M_x - M_{x+n} + D_{x+n}) \mp d \cdot [S_{x+1} - S_{x+n} - (n-1) \cdot N_{x+n}]}{N_x - N_{x+n}}$$

Relation (5)

ii. If the premium is behaving in geometric progression:

$$\ddot{A}_{xn]} \cdot l_x = R_1 \cdot (l_x + q \cdot l_{x+1} \cdot v + q^2 \cdot l_{x+2} \cdot v^2 + q^3 \cdot l_{x+3} \cdot v^3 + \cdots + q^{n-1} \cdot l_{x+n-1} \cdot v^{n-1})$$

After appropriate mathematical ordering relation we get for this model of insurance is as follows:

$$\ddot{A}_{xn]} \cdot D_x = R_1 \cdot (D_x + q \cdot D_{x+1} + q^2 \cdot D_{x+2} + q^3 \cdot D_{x+3} + \cdots + q^{n-1} \cdot D_{x+n-1})$$

On the basis of mutual relation between commutation symbols we get the following final relation for this model of insurance:

$$\ddot{A}_{xn]} = \frac{R_1 \cdot (N'_x - N'_{x+n})}{D_x}$$

Applying the principle of equivalence, we get the final expression for calculating the amount of premiums in the model which has the following characteristics: multiple premiums are paid and in yearly intervals, at the beginning of each year, throughout the duration of contract relations; the premium varies throughout the obligations and in a way that in each successive period premium is higher or lower than premiums in the previous period for the same relative amount and they provide a lump sum of capital to be paid at the end of the year in which the insured person dies or at the end of the year in which the insured survives age $x+n$. Defined premium, according to the principle of equivalence, equate with the withdrawals so the expression follows:

$$\frac{P_1 \cdot (N'_x - N'_{x+n})}{D_x} = K \cdot \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

$$P_1 = \frac{K \cdot (M_x - M_{x+n} + D_{x+n})}{N'_x - N'_{x+n}}$$

Relation (6)

d) If premiums are $P^{(m)}_{(1)}(EA_{xn]})$, then the principle of equivalence of deposits and withdrawals, in this sub model will look like this:

i. If the premium is behaving in arithmetic progression:

Actuarial analogy shown in previous models, it is possible to quantify the risks for all other models, therefore detailed performance of model, which is shown below will not be elaborated. This model is, by its characteristics, very similar to the model shown in relation (5). Premiums are paid multiple times and in m intervals during one year, at the beginning of each period year, throughout the duration of contract relations; the premium varies throughout the obligations but not the way it is elaborated in relation (5) (in this case, the premium are the same in the course of one year and at the beginning of each subsequent year will increase by the same amount); premiums provide one capital payment to be paid at the end of the year in which the insured person dies or at the end of the year in which the insured survives age $x+n$.

$$P_1 \cdot \left(m \cdot \frac{N_x - N_{x+n}}{D_x} + \frac{m-1}{2} \cdot \frac{D_{x+n}}{D_x} - \frac{m-1}{2}\right) \pm$$

$$\pm d \cdot \left[m \cdot \frac{S_{x+1} - S_{x+n}}{D_x} - \frac{m-1}{2} \cdot \frac{N_{x+1}}{D_x} + \frac{m-1-2 \cdot m \cdot (n-1)}{2} \cdot \frac{N_{x+n}}{D_x} + \frac{m-1}{2} \cdot (n-1) \cdot \frac{D_{x+n}}{D_x}\right] = K \cdot \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

$$P_1 = \frac{K \cdot \frac{M_x - M_{x+n} + D_{x+n}}{D_x}}{m \cdot \frac{N_x - N_{x+n}}{D_x} + \frac{m-1}{2} \cdot \frac{D_{x+n}}{D_x} - \frac{m-1}{2}} \mp$$

$$\mp \frac{d \cdot \left[m \cdot \frac{S_{x+1} - S_{x+n}}{D_x} - \frac{m-1}{2} \cdot \frac{N_{x+1}}{D_x} + \frac{m-1-2 \cdot m \cdot (n-1)}{2} \cdot \frac{N_{x+n}}{D_x} + \frac{m-1}{2} \cdot (n-1) \cdot \frac{D_{x+n}}{D_x}\right]}{m \cdot \frac{N_x - N_{x+n}}{D_x} + \frac{m-1}{2} \cdot \frac{D_{x+n}}{D_x} - \frac{m-1}{2}}$$

Relation (7)

ii. If the premium is behaving in a geometric progression:

$$P_1 \cdot \left[\frac{m \cdot (N'_x - N'_{x+n}) + \frac{m-1}{2} \cdot q^n \cdot D_{x+n}}{D_x} - \frac{m-1}{2}\right] = K \cdot \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

$$P_1 = \frac{K \cdot \frac{M_x - M_{x+n} + D_{x+n}}{D_x}}{\frac{m \cdot (N'_x - N'_{x+n}) + \frac{m-1}{2} \cdot q^n \cdot D_{x+n}}{D_x} - \frac{m-1}{2}}$$

Relation (8)

Ad 2.Sub model, where the period of premium payments are as long as obligation, and capital payment period is shorter than the obligation period, may be determined as follows:

Person age x years wants to provide a one-time payment of amount K in case of survival for next n years or in case of earlier death. Premium is being paid during the entire period of contract, and payment of the insured capital begins k years after the obligation and lasts until the end of contract. What is the net premium for this insurance plan?

$P_1, P_2, P_3, P_4, \dots, P_n$ – premiums that can be:

$P(EA_{xn]})$ and /or $P^{(m)}(EA_{xn]})$ and /or $P_{(1)}(EA_{xn]})$ and /or $P_{(1)}^{(m)}(EA_{xn]})$

In this sub model possible forms of premiums payment are the same as in the previous sub model presented by Ad 1. In this period sub model period for payment of the insured capital is shorter and lasts for k years. On the mathematical principle of equivalence, it means that for the sub model (2) it is possible to use the left side of the initial equation of all sub models listed at (1), and the right side of the starting equation reads (for all observed forms of payment of premium):

$$K \cdot \frac{M_{x+k} - M_{x+k+n} + D_{x+k+n}}{D_x}$$

Relation (9)

Ad 3.Sub model, where the premium payment period is shorter than the period of contract, and payment period of the insured capital is equal to the period of obligations, can be determined as follows:

Person age x wants to provide a one-time payment of amount K in case of survival for next n years or in case of earlier death, if the premiums are being paid during the n_u years of contract period $(x + n_u)$. What is the net premium for this insurance plan?

$P_1, P_2, P_3, P_4, \dots, P_n$ – premiums that can be:

$P(EA_{xn]})$ and /or $P^{(m)}(EA_{xn]})$ and /or $P_{(1)}(EA_{xn]})$ and /or $P_{(1)}^{(m)}(EA_{xn]})$

In this sub model possible forms of payment can have the same essence and form as in the Ad 1, but their period of payment is shorter than obligation period (denoted with symbol n_u). With the mathematical principle of equivalence it means that, compared to the Ad 1, at the Ad 3 the left side of the starting equation needs corrections, while the right side of this equation remains the same. So, on the left side of the initial equation, based on which they later explicitly expressed by the equations: (3), (4), (5), (6), (7) and (8) instead of the symbol n we use symbol n_u, where in the sub model (3) it is clearly specified that $n_u < n$.

Ad 4.Sub model where the period of premium payment and period of insured capital payment are shorter than the period of contract, and payment period of the insured capital directly follows the period of premium payment. Model can be determined as follows:

Person age x wants to provide a one-time payment of amount K in case of survival for next n years or in case of earlier death, if the premiums are being paid during the n_u years of contract period$x + n_u$, and payment of the insured capital is made in period $x + n - n_u$, which means immediately after last period of paying premiums until the end of contract period . What is the net premium for this insurance plan?

$P_1, P_2, P_3, P_4, \dots, P_n$ – premiums that can be:

$P(EA_{xn]})$ and /or $P^{(m)}(EA_{xn]})$ and /or $P_{(1)}(EA_{xn]})$ and /or $P_{(1)}^{(m)}(EA_{xn]})$

Sub model Ad 4 is a combination of Ad 3 (from the standpoint of paying premiums) and Ad 2 (from the standpoint of the capital payment, provided that it is $x + n_u = x + k$.

Ad 5.Sub model where the period of payment of the insured capital just does not continue for a period of premium payments can be determined as follows:

Person age x wants to provide a one-time payment of amount K in case of survival for next n years or in case of earlier death, if the premiums are being paid during n_u years of contract period, and payment of the insured capital cannot directly follow a period of premium payments, but made it two or more years after stopping payment of premiums. What is the net premium for this insurance plan?

$P_1, P_2, P_3, P_4, \dots, P_n$ – premiums that can be:

$P(EA_{xn]})$ and /or $P^{(m)}(EA_{xn]})$ and /or $P_{(1)}(EA_{xn]})$ and /or $P_{(1)}^{(m)}(EA_{xn]})$

Sub model Ad 5 is just a specific form of sub model Ad 4. In sub model Ad 4 period of payment of the insured capital directly continues the past period of premium payments and in sub model Ad 5 there is an appropriate time interval, a minimum of two years (or periods) between the end of the last payment of premiums and the first disbursement of the insured capital.

All forms, as a result of the principle of equivalence of premium payment and the payment of the insured capital, which are valid for sub model Ad 4 could be applied to the sub model Ad 5, provided that the period of paying premiums lasts from x to $x + n_u$, a period of disbursement of the insured amount lasts from $x + n_i$ to $x + n$, where $n_i \geq 2$.

Financial characteristics of the capital endowment insurance model

All models and sub models of life insurance have their own specific financial characteristics. They have common characteristics, but also the individual ones. Common characteristics are the primary determinants of cash flow: inflow (income) and outflows (expenses), as well as a peremptory need for proper placement of temporary free funds. The actual inflow of funds, with the character of the net premiums, until their final outflow from the account of the insurer, based on the contract with the policyholder, have the characteristics of temporarily free funds. That means that according to the laws of actuarial mathematics, those funds should be qualified for a certain time in order to be adequately increased. In essence, it is respect for the legality of the time value of money and stochastic processes incorporated in the algorithms of actuarial mathematics and financial management.

Models and sub models of endowment insurance with their financial characteristics have more parameters and variables that directly affect them. Parameters are related to the characteristics of sub models. In the first place they are related to the time of premiums payment and the time of disbursement of the insured capital. The variables are directly present in the modalities of premiums payment and in the desired (possible) amount of insured capital, and indirectly in the calculation of commutation symbols. Parameters and variables are linked to instruments where the investments of temporarily free funds will be made (synthetically said: financial market and the market in real assets, financial institutions, etc.).

Financial characteristics of the model and the sub models of endowment insurance from the perspective of participants in the obligation, in principle, have different starting points and the same goal. The starting points are, in principle, different from the standpoint of the number and manner of premiums payment and the aim is tentatively the same: the real achievement of its objectives within the agreed contractual relations.

Sub models presented in this paper show that the premium can be paid once and repeatedly (with a range of modalities). All these modes of payment have a corresponding advantages and disadvantages. From the standpoint of the policyholder are acceptable forms of multiple premium payments, while from the standpoint of insurers (insurance company) form of one-time payment is preferred. It is natural that by paying a premium multiple contract parties agreed modalities of payment: in one or more series and with the modalities presented in the previous section of this paper.

From the viewpoint of insurers, the most acceptable way of premium payment is single premium in the case where a capital payment is deferred comparing to start of obligation or single premium payment. In this case the insurer has the largest choice of funds' investment. All other sub models of endowment insurance have worse financial repercussions for the insurer (normally, if the start point is comparative baseline size). The general principle is clear: the more the funds and the more money have status of temporarily free funds, i.e. they are more available to the insurer, the greater the possibility of their investment is. The higher the yield on insurers' provisions is in the interest of all parties in the contract and has synergetic effects.

Of course, it should be noted that, with the actuarial point of view, it is possible to change each of the defined parameters, so in that way to recognizes a large number of possible models. However, from a practical point of view, all such models are not acceptable to modern business practices. Premiums in actuarial practice can be regarded as temporary or permanent category; a start may be immediately after signing the contract or later in relation to the contract start. In the models listed here, permanently premium payment is usually not used in practice (and would not have any real meaning). The premium can be determined as a fixed or variable (where variability implies any subsequent exactly the premium - namely, the existence of mathematical principle). In addition, the premium can be defined as the pure premium, which would imply that the premium is calculated individually for each subsequent year. Defined premiums would have an upward trend that was due to increased mortality in actuarial tables used by insurance company, and would reflect, quite precisely, the risk of death in each year. However, from a practical point of view, such a premium would not be really useful and in practice it is generally not used.[6]

CONCLUSION

Insurance of fixed capital in case of survival or early death - capital endowment insurance is the most common form of life insurance with lump-sum payments. This actuarial product (service) is particularly interesting for insurance customers (insurance beneficiaries), and of particular interesting to insurers. Basic models of this insurance plan, with its many sub models provide a wide possibility of arbitration, both from the standpoint of the insured (the policyholder) and from the standpoint of insurers. Many sub models provide a reasonable basis to find common acceptable (and desirable) solutions in these potential contractual relations.

Actuarial science, actuarial mathematics and modern financial management have enabled the scientific and technical merits of the profitability of this product of the insurance industry. Contemporary forms of insurance company financial intermediation in the financial markets and real assets' markets are the real basis for multiple positive effects for all parties in contract. The insured in this way gets the necessary basis for the additional financial, material security, while the insurer gets the additional option not only for their primary business - fulfillment form of life insurance, but for an active institutional investor as well. The presented models and sub models clearly prove the mentioned statements.

APPENDIX I: Notation

Symbol	Meaning
x	Policyholder's age (in years)
l_x	Number of living persons of age x
d_x	Number of death persons of age x
n	Duration of obligation (contract) in years
n_u	Duration of premiums payment (years)
n_i	Duration of capital payment (years)
k	Years of capital payment postponement from the start of the contract
K	Nominal amount of lump sum (capital) payment (insured sum)
K_1	Nominal amount of lump sum (capital) payment if the payment is done in the 1st yr
$EA_{xn\rceil}$	Single pure premium
$P(EA_{xn\rceil})$	Multiple annual pure premium
$P^{(m)}(EA_{xn\rceil})$	Pure premium that is paid in m periods in year
$P_{(1)}(EA_{xn\rceil})$	First annual variable pure premium
$P_{(1)}^{(m)}(EA_{xn\rceil})$	First variable pure premium that is paid in m periods in year
m	Number of periods in one year
d	Difference in arithmetic progression
s	Rate of change in geometric progression
$N'_x = D_x + qD_{x+1} + q^2D_{x+2} + \cdots$	and $N'_{x+n} = q^nD_{x+n} + q^{n+1}D_{x+n+1} + \cdots$

[6] Pure premium is the most precise price for the risk of death, but the insurance companies it only appears as an actuarial category. In practice is not in use because it would mean that the insured with increasing age should pay a higher premium amount, which certainly is not rational from the financial point of view.

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